

A Monotone Scheme for High Dimensional Fully Nonlinear PDEs

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Abstract

In this paper we propose a monotone numerical scheme for fully nonlinear parabolic PDEs, which includes the quasi-linear PDE associated with a coupled FBSDE as a special case. Our paper is strongly motivated by the remarkable work Fahim, Touzi and Warin [15], and stays in the paradigm of monotone schemes initiated by Barles and Souganidis [3]. Our scheme weakens a critical constraint imposed by [15], especially when the generator of the PDE depends only on the diagonal terms of the hessian matrix. Several numerical examples, up to dimension 12, are reported.

Key words: Monotone scheme, Monte Carlo methods, fully nonlinear PDEs, viscosity solutions

AMS 2010 subject classifications: 65C05; 49L25

1 Introduction

In this paper we are interested in feasible numerical schemes for the following fully nonlinear parabolic PDE, especially in high dimensional case:

$$-\partial_t u - G(t, x, u, Du, D^2u) = 0 \text{ on } [0, T) \times \mathbb{R}^d; \quad u(T, \cdot) = g(\cdot) \text{ on } \mathbb{R}^d. \quad (1.1)$$

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The standard numerical schemes in the PDE literature, e.g. finite difference method and finite elements method, work only for low dimensional problems, typically $d \leq 3$, due to the well known *curse of dimensionality*. However, in many applications, especially in finance, the dimension d can be higher. We thus turn to probabilistic approach which is less sensitive to the dimension.

In the semilinear case, the PDE (1.1) is associated to a Markovian Backward SDE due to the nonlinear Feynman-Kac formula introduced by Pardoux and Peng [23]. Based on the regularity results of BSDEs established by Zhang [28], [28] and Bouchard and Touzi [8] proposed the so called Backward Euler Scheme for such BSDEs and hence for the associated semilinear PDEs, and obtained the rate of convergence. This scheme approximates the BSDE by a sequence of conditional expectations, and several efficient numerical algorithms have been proposed to compute these conditional expectations, notably: Bouchard and Touzi [8], Gobet-Lemor-Waxin [17], Bally-Pages-Printems [1]), Bender-Denk [5], Crisan-Manolarakis [12]. There have been numerous publications on the subject and the schemes have been extended to more general BSDEs, e.g. reflected BSDEs which is appropriate for pricing and hedging American options. Typically these algorithms work for 10 or higher dimensional problems.

We intend to numerically solve PDE (1.1) in fully nonlinear case, in particular the Hamilton-Jacobi-Bellman equations and the Bellman-Isaacs equations which are widely used in stochastic control and in stochastic differential games. We remark that this is actually one main motivation of the developments of second order BSDEs by Cheridito, Soner, Touzi and Victoir [9], and Soner, Touzi and Zhang [25]. Our scheme is strongly inspired by the work Fahim, Touzi and Warin [15]. Based on the monotone scheme of Barles and Souganidis [3], [15] extended the backward Euler scheme to fully nonlinear PDE (1.1). In the case G is convex in (u, Du, D^2u) , they obtained the rate of convergence by using the techniques in Krylov [18] and Barles and Jakobson [2]. They applied the linear regression method, see e.g. [17], to compute the involved conditional expectations, and presented some numerical examples up to dimension 5. We remark that the rate of convergence has been improved very recently by Tan [26], by using purely probabilistic arguments.

There is one critical constraint in [15] though. In order to ensure the monotonicity of the backward Euler scheme, they assume the lower and upper bounds of G_γ , the derivative of G with respect to D^2u , satisfies certain constraint. However, when the dimension is high, this constraint implies that G_γ is essentially a constant and thus the PDE (1.1) is essentially semilinear, see (2.8) for more details. This is of course not desirable in practice.

The main contribution of this paper is to relax the above constraint. In [15] and most papers in the literature of numerical BSDEs, the involved

conditional expectations are expressed in terms of Brownian motion. Our first simple but important observation is that we may replace the unbounded Brownian motion with bounded trinomial tree, which helps to maintain the monotonicity of the scheme. We next modify the scheme further, but still in the paradigm of monotone scheme, so as to relax the constraint. In the special case where G_γ is diagonal, namely G involves D^2u only through its diagonal terms, the above constraint is removed completely. Rate of convergence is also obtained. Several numerical examples are presented. In low dimensional case, our scheme is comparable to finite difference method and is superior to the simulation methods. When G_γ is diagonal, our scheme works well for 12 dimensional problems.

We note that PDE (1.1) covers the quasilinear PDEs as a special case, which corresponds to a coupled forward backward SDE due to the four step scheme of Ma, Protter and Yong [19]. There are only a few papers on numerical methods for FBSDEs, e.g. Douglas-Ma-Protter [19], Makarov [21], Cvitanic and Zhang [11], Delarue and Menozzi [13], Milstein-Tretyakov [22], Bender and Zhang [7], and Ma, Shen and Zhao [20]. Most of them deal with low dimensional FBSDEs only, except that [7] reported a 10-dimensional numerical example. However, [7] proved the rate of convergence only for time discretization, and the convergence of the linear regression approximation is not analyzed theoretically. Our scheme works for FBSDEs as well, especially when the diffusion coefficient σ is diagonal. A numerical example for a 12-dimensional coupled FBSDE is reported.

We have also presented a few numerical examples which violate our assumptions and thus the scheme may not be monotone. Numerical results show that our scheme still converges. In particular, we note that our current theoretical result does not cover the G -expectation, a nonlinear expectation introduced by Peng [24]. We nevertheless implement our algorithm to approximate a 10-dimensional HJB equation, which includes the G -expectation as a special case, and it indeed converges to the true solution. It will be very interesting to investigate the convergence of our scheme, or its variations if necessary, when the monotonicity condition is violated. We shall leave this for our future research.

The rest of the paper is organized as follows. In section 2 we present some preliminaries. In Section 3 we propose our scheme and prove the main convergence results. Section 4 is devoted to the study of quasilinear PDEs and the associated coupled FBSDEs. In Section 5 we discuss how to approximate the involved conditional expectations. Finally we present several numerical examples in Section 6, up to dimension 12.

2 Preliminaries

Let $T > 0$ be the terminal time, $d \geq 1$ the dimension of the state variable x , \mathbb{S}^d the set of $d \times d$ symmetric matrices. For $x, \tilde{x} \in \mathbb{R}^d$ and $\gamma, \tilde{\gamma} \in \mathbb{S}^d$, denote

$$x \cdot \tilde{x} := \sum_{i=1}^d x_i \tilde{x}_i, \quad |x| := \sqrt{x \cdot x}, \quad \text{and} \quad \gamma : \tilde{\gamma} := \text{tr}(\gamma \tilde{\gamma}), \quad |\gamma| := \sqrt{\gamma : \gamma}.$$

For $x \in \mathbb{R}^d$, x^T denotes its transpose, and thus $xx^T \in \mathbb{S}^d$. For $\gamma, \tilde{\gamma} \in \mathbb{S}^d$, we say $\gamma \leq \tilde{\gamma}$ if $\tilde{\gamma} - \gamma$ is nonnegative definite. For any $\gamma = [\gamma_{ij}] \in \mathbb{S}^d$, denote

$$D[\gamma] := \text{the diagonal matrix whose } (i, i)\text{-th component is } \gamma_{ii}. \quad (2.1)$$

It is clear that, for any $\gamma, \tilde{\gamma} \in \mathbb{S}^d$,

$$D[\gamma] : \tilde{\gamma} = D[\gamma] : D[\tilde{\gamma}] = \gamma : D[\tilde{\gamma}]. \quad (2.2)$$

Moreover, we use the same notation $\mathbf{0}$ to denote the zeroes in \mathbb{R}^d and \mathbb{S}^d .

Our objective is the PDE (1.1), where $G : (t, x, y, z, \gamma) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d \rightarrow \mathbb{R}$ and $g : x \in \mathbb{R}^d \rightarrow \mathbb{R}$. We shall always assume the following standing assumptions:

Assumption 2.1. (i) $G(\cdot, 0, \mathbf{0}, \mathbf{0})$ and g are bounded.

(ii) G is continuous in t , uniformly Lipschitz continuous in (x, y, z, γ) , and g is uniformly Lipschitz continuous in x

(iii) PDE (1.1) is parabolic, that is, G is nondecreasing in γ .

(iv) The PDE (1.1) has a unique bounded viscosity solution u , and the comparison principle for its bounded viscosity solutions holds true.

For notational simplicity, throughout the paper we assume further that

G is differentiable in (y, z, γ) so that we can use the notations G_γ etc.

However, we emphasize that all the results in the paper do not rely on this additional assumption. For the theory of viscosity solutions, we refer to the classical references Crandall, Ishii and Lions [10] and Fleming and Soner [16]. Our goal of the paper is to numerically compute the viscosity solution u . In their seminal work Barles and Souganidis [3] proposed a monotone scheme in an abstract way and proved its convergence by using the viscosity solution approach. To be precise, for any $t \in [0, T)$ and $h > 0$ with $t + h \leq T$, let \mathbb{T}_h^t be an operator on the set of measurable functions $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$. Now $n \geq 1$, denote $h := \frac{T}{n}$, $t_i := ih$, $i = 0, 1, \dots, n$, and define:

$$u_h(t_n, x) := g(x), \quad u_h(t, \cdot) := \mathbb{T}_{t_i - t}^t[u_h(t_i, \cdot)], \quad t \in [t_{i-1}, t_i], \quad i = n, \dots, 1. \quad (2.3)$$

The following convergence result is due to Fahim, Touzi and Warin [15] Theorem 3.6, which is based on [3].

Theorem 2.2. *Let Assumption 2.1 hold. Assume the operator \mathbb{T}_h^t satisfies the following four conditions:*

(i) *Consistency: for any $(t, x) \in [0, T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0, T) \times \mathbb{R}^d)$,*

$$\lim_{(t', x', h, c) \rightarrow (t, x, 0, 0)} \frac{[c + \varphi](t', x') - \mathbb{T}_h^t[[c + \varphi](t', \cdot)](x')}{h} = -\partial_t \varphi(t, x) - G(t, x, \varphi, D\varphi, D^2\varphi).$$

(ii) *Monotonicity: $\mathbb{T}_h^t[\varphi] \leq \mathbb{T}_h^t[\psi]$ whenever $\varphi \leq \psi$.*

(iii) *Stability: u_h is bounded uniformly in h whenever g is bounded.*

(iv) *Boundary condition: for any $x \in \mathbb{R}^d$,*

$$\liminf_{(t', x', h) \rightarrow (T, x, 0)} u_h(t', x') = \limsup_{(t', x', h) \rightarrow (T, x, 0)} u_h(t', x').$$

Then u_h converges to u locally uniformly as $h \rightarrow 0$.

[15] proposed a scheme $\bar{\mathbb{T}}_h^t$ as follows. Assume there exist $\underline{\sigma}, \bar{\sigma} \in \mathbb{S}^d$ such that $\mathbf{0} < \frac{1}{2}\underline{\sigma}^2 \leq G_\gamma \leq \frac{1}{2}\bar{\sigma}^2$. Denote

$$\bar{F}(t, x, y, z, \gamma) := G(t, x, y, z, \gamma) - \frac{1}{2}\underline{\sigma}^2 : \gamma, \quad (2.4)$$

and define

$$\bar{\mathbb{T}}_h^t[\varphi](x) := \bar{\mathcal{D}}_h^0 \varphi(x) + h \bar{F}\left(t, x, \bar{\mathcal{D}}_h^0 \varphi(x), \bar{\mathcal{D}}_h^1 \varphi(x), \bar{\mathcal{D}}_h^2 \varphi(x)\right), \quad (2.5)$$

where, for a d -dimensional standard Normal random variable N ,

$$\bar{\mathcal{D}}_h^i \varphi(x) := \mathbb{E}\left[\varphi(x + \sqrt{h}\underline{\sigma}N)\bar{K}_i(N)\right], \quad i = 0, 1, 2, \quad (2.6)$$

$$\bar{K}_0(N) := 1, \quad \bar{K}_1(N) := \frac{\underline{\sigma}^{-1}N}{\sqrt{h}}, \quad \bar{K}_2(N) := \frac{\underline{\sigma}^{-1}[NN^T - I_d]\underline{\sigma}^{-1}}{h}.$$

This scheme satisfies the consistency, and the stability follows from the monotonicity. However, to ensure the monotonicity, one needs to assume $F_\gamma : \underline{\sigma}^{-2} \leq 1$, see the proof of [15] Lemma 3.12. This essentially requires

$$\left[\frac{1}{2}\bar{\sigma}^2 - \frac{1}{2}\underline{\sigma}^2\right] : \underline{\sigma}^{-2} \leq 1, \quad \text{and thus} \quad \bar{\sigma}^2 : \underline{\sigma}^{-2} \leq d + 2. \quad (2.7)$$

In the case $\frac{1}{2}\underline{\sigma}^2 = \underline{\alpha}I_d, \frac{1}{2}\bar{\sigma}^2 = \bar{\alpha}I_d$, we have

$$1 \leq \bar{\alpha}/\underline{\alpha} \leq 1 + \frac{2}{d}. \quad (2.8)$$

When d is large, this implies $\bar{\alpha} \approx \underline{\alpha}$ and thus G is essentially semilinear, which of course is not desirable in practice.

Our goal of this paper is to modify the algorithm (2.5)-(2.6) so as to relax the above constraint. In particular, in the case that G_γ is diagonal, we remove this constraint completely.

3 The numerical scheme

In this section we present our numerical scheme and study its convergence. Our scheme involves two parameters $\mathbf{0} < \sigma_0 \in \mathbb{S}^d$ and $0 < p < 1$, which will be specified in Section 3.2 below. We shall always denote

$$F(t, x, y, z, \gamma) := G(t, x, y, z, \gamma) - \frac{1}{2}\sigma_0^2 : \gamma, \quad \tilde{G}_\gamma := \sigma_0^{-1}G_\gamma\sigma_0^{-1}. \quad (3.1)$$

However, unlike in [15], we emphasize that we do not require $\frac{1}{2}\sigma_0^2 \leq G_\gamma$. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, and $\xi : \Omega \rightarrow \mathbb{R}^d$ be a random variable such that its components ξ_i , $i = 1, \dots, d$ are independent and have the identical distribution:

$$\mathbb{P}(\xi_i = \frac{1}{\sqrt{p}}) = \frac{p}{2}, \quad \mathbb{P}(\xi_i = -\frac{1}{\sqrt{p}}) = \frac{p}{2}, \quad \mathbb{P}(\xi_i = 0) = 1 - p. \quad (3.2)$$

This implies that

$$\mathbb{E}[\xi_i] = \mathbb{E}[\xi_i^3] = 0, \quad \mathbb{E}[\xi_i^2] = 1, \quad \mathbb{E}[\xi_i^4] = \frac{1}{p}. \quad (3.3)$$

We now modify the algorithm (2.5)-(2.6):

$$\mathbb{T}_h^t[\varphi](x) := \mathcal{D}_h^0\varphi(x) + hF\left(t, x, \mathcal{D}_h^0\varphi(x), \mathcal{D}_h^1\varphi(x), \mathcal{D}_h^2\varphi(x)\right), \quad (3.4)$$

where, recalling (2.1),

$$\begin{aligned} \mathcal{D}_h^i\varphi(x) &:= \mathbb{E}\left[\varphi(x + \sqrt{h}\sigma_0\xi)K_i(\xi)\right], \quad i = 0, 1, 2, \\ K_0(\xi) &:= 1, K_1(\xi) := \frac{\sigma_0^{-1}\xi}{\sqrt{h}}, K_2(\xi) := \frac{\sigma_0^{-1}\left[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d\right]\sigma_0^{-1}}{(1-p)h}. \end{aligned} \quad (3.5)$$

One may check straightforwardly that

$$E[K_1(\xi)] = \mathbf{0}, \quad E[K_2(\xi)] = \mathbf{0}. \quad (3.6)$$

We recall that the approximating solution u_h is defined by (2.3).

Remark 3.1. *If we assume $\frac{1}{2}\sigma_0^2 \leq G_\gamma$ and set $p = \frac{1}{3}$, then our scheme is obtained by replacing the normal random variable N in (2.6) with trinomial random variable ξ . This in fact has already been mentioned in [15].*

3.1 Consistency

We first justify our scheme by checking its consistency.

Lemma 3.2. *Under Assumption 2.1, \mathbb{T}_h^t satisfies the consistency requirement in Theorem 2.2 (i).*

Proof. We first assume $(t', x', c) = (t, x, 0)$ and send $h \downarrow 0$. Let $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$. Apply the Taylor expansion we have

$$\begin{aligned} \varphi(t+h, x+\sqrt{h}\sigma_0\xi) &= \varphi(t, x) + \partial_t\varphi(t, x)h + \sqrt{h}D\varphi(t, x) \cdot \sigma_0\xi \\ &\quad + \frac{h}{2}D^2\varphi(t, x) : [\sigma_0\xi][\sigma_0\xi]^T + o(h). \end{aligned}$$

By (3.3) and the independence of ξ_k one may check straightforwardly that

$$\begin{aligned} \mathcal{D}_h^0\varphi(t+h, \cdot)(x) &= \varphi(t, x) + \partial_t\varphi(t, x)h + \frac{h}{2}D^2\varphi(t, x) : \sigma_0^2 + o(h); \\ \mathcal{D}_h^1\varphi(t+h, \cdot)(x) &= D\varphi(t, x) + o(\sqrt{h}). \end{aligned} \tag{3.7}$$

Moreover, for any $i \neq j$,

$$\begin{aligned} \mathbb{E}\left[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d\right] &= \mathbf{0}; \\ \mathbb{E}\left[\xi_i[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\right] &= \mathbf{0}; \\ \mathbb{E}\left[\xi_i^2[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\right] &= 2(1-p)\delta_{i,i}; \\ \mathbb{E}\left[\xi_i\xi_j[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\right] &= (1-p)(\delta_{i,j} + \delta_{j,i}) \end{aligned}$$

Here $\delta_{i,j}$ is the $d \times d$ -matrix whose (i, j) -th component is 1 and all other components are 0. Then, denoting $A = [a_{i,j}] := \sigma_0 D^2\varphi(t, x)\sigma_0$,

$$\begin{aligned} &\mathbb{E}\left[(D^2\varphi(t, x) : [\sigma_0\xi][\sigma_0\xi]^T)\sigma_0^{-1}[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\sigma_0^{-1}\right] \\ &= \sigma_0^{-1}\mathbb{E}\left[(A : \xi\xi^T)[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\sigma_0^{-1}\right] \\ &= \sigma_0^{-1}\sum_{i,j=1}^d a_{i,j}\mathbb{E}\left[\xi_i\xi_j[(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d]\right]\sigma_0^{-1} \\ &= (1-p)\sigma_0^{-1}\sum_{i,j=1}^d a_{i,j}(\delta_{i,j} + \delta_{j,i})\sigma_0^{-1} \\ &= 2(1-p)\sigma_0^{-1}A\sigma_0^{-1} = 2(1-p)D^2\varphi(t, x), \end{aligned}$$

and thus

$$\mathcal{D}_h^2 \varphi(t+h, \cdot)(x) = D^2 \varphi(t, x) + o(1). \quad (3.8)$$

Combine (3.7) and (3.8), one sees immediately that

$$\lim_{h \downarrow 0} \frac{\varphi(t, x) - \mathbb{T}_h^t[\varphi(t, \cdot)](x)}{h} = -\partial_t \varphi(t, x) - G(t, x, \varphi, D\varphi, D^2\varphi).$$

The general consistency follows from straightforward extension of the above arguments and we omit the details. \square

3.2 The monotonicity

To obtain the monotonicity of our scheme, we need to impose the following key assumption of the paper.

Assumption 3.3. *There exist $0 < \sigma_0 \in \mathcal{S}^d$, scalar constants $0 < \underline{\alpha} < \bar{\alpha}$ and $0 \leq \theta < \frac{2}{d}$ such that*

- (i) $\underline{\alpha} I_d \leq D[\tilde{G}_\gamma] \leq \bar{\alpha} I_d$ and $D[\tilde{G}_\gamma] \leq (1 + \theta) \tilde{G}_\gamma$;
- (ii) $\Lambda := \bar{\alpha} / \underline{\alpha} < \Lambda_\theta := \sup_{\frac{\theta}{2(1+\theta)} \leq p \leq \frac{1}{3}} \lambda(p, \theta)$, where

$$\lambda(p, \theta) := 1 + \frac{2p - \theta - p(d-3)\theta}{2p^2(1+\theta)(d-1)}. \quad (3.9)$$

(iii) By otherwise rescaling σ_0 (namely multiplying σ_0 by a scalar constant) we may set without loss of generality that

$$\begin{aligned} p &\in [\frac{\theta}{2(1+\theta)}, \frac{1}{3}] \text{ such that } \Lambda < \lambda(p, \theta) \leq \Lambda_\theta; \\ \bar{\alpha} &= \frac{\Lambda}{2p(\Lambda-1) + \alpha_p} \text{ where } \alpha_p := \frac{p(2+3\theta) - \theta}{p(1+\theta)}. \end{aligned} \quad (3.10)$$

Remark 3.4. In this remark we provide a few facts concerning our choices of parameters which will be used in the proof of next lemma.

(i) We need $p \leq \frac{1}{3}$ so that $1 - 3p$, the coefficient of $D[\xi \xi^T]$, is nonnegative. For $0 \leq \theta < \frac{2}{d}$, we have $\frac{\theta}{2(1+\theta)} < \frac{1}{3}$. Moreover, for $p \in [\frac{\theta}{2(1+\theta)}, \frac{1}{3}]$, it holds that $\alpha_p \geq 2p$ and $\lambda(p, \theta) > 1$.

(ii) Note that Λ is invariant of the scaling of σ_0 and recall that $\underline{\alpha} = \bar{\alpha} / \Lambda$. Then our choice of $\bar{\alpha}$ implies $2\bar{\alpha} = \frac{1}{p} + (2 - \frac{\alpha_p}{p})\underline{\alpha}$. \square

Lemma 3.5. *Let Assumptions 2.1 and 3.3 hold. Then there exists a constant $h_0 > 0$, depending on $d, T, \theta, p, \Lambda, \lambda(p, \theta)$, and the bound and Lipschitz constants in Assumption 2.1, such that \mathbb{T}_h^t satisfies the monotonicity in Theorem 2.2 (ii) for all $h \in (0, h_0]$.*

Proof. Let $\varphi_1 \leq \varphi_2$ be bounded and $\psi := \varphi_2 - \varphi_1 \geq 0$. Then by (3.4) we have, at (t, x) ,

$$\mathbb{T}_h[\varphi_2] - \mathbb{T}_h[\varphi_1] = \mathcal{D}_h^0 \psi + h \left[F_y \mathcal{D}_h^0 \psi + F_z \cdot \mathcal{D}_h^1 \psi + F_\gamma : \mathcal{D}_h^2 \psi \right]. \quad (3.11)$$

Here the terms F_y, F_z, F_γ are defined in an obvious way and we emphasize that they are deterministic. Plug (3.5) into above equality, then

$$\begin{aligned} \mathbb{T}_h[\varphi_2] - \mathbb{T}_h[\varphi_1] &= \mathbb{E} \left[\psi(x + \sqrt{h} \sigma_0 \xi) \left[1 + h[F_y + F_z \cdot K_1(\xi) + F_\gamma : K_2(\xi)] \right] \right] \\ &= \mathbb{E} \left[\psi(x + \sqrt{h} \sigma_0 \xi) \left[1 + hF_y + \sqrt{h} F_z \cdot (\sigma_0^{-1} \xi) + \frac{I}{1-p} \right] \right] \\ &\geq \mathbb{E} \left[\psi(x + \sqrt{h} \sigma_0 \xi) \left[1 - C\sqrt{h} + \frac{I}{1-p} \right] \right], \end{aligned} \quad (3.12)$$

thanks to Assumption 2.1, where,

$$\begin{aligned} I &:= F_\gamma : \left(\sigma_0^{-1} [(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d] \sigma_0^{-1} \right) \\ &= [\tilde{G}_\gamma - \frac{1}{2}I_d] : [(1-p)\xi\xi^T - (1-3p)D[\xi\xi^T] - 2pI_d] \\ &= (1-p)[\tilde{G}_\gamma - \frac{1}{2}I_d] : (\xi\xi^T) - (1-3p)[D[\tilde{G}_\gamma] - \frac{1}{2}I_d] : (\xi\xi^T) \\ &\quad - 2p\text{tr}(\tilde{G}_\gamma) + pd. \end{aligned}$$

Denote $\alpha_i := (\tilde{G}_\gamma)_{ii}$. Then it follows from Assumption 3.3 (i) that

$$\begin{aligned} I &\geq (1-p) \left[\frac{1}{1+\theta} D[\tilde{G}_\gamma] - \frac{1}{2}I_d \right] : (\xi\xi^T) - (1-3p) \left[D[\tilde{G}_\gamma] - \frac{1}{2}I_d \right] : (\xi\xi^T) \\ &\quad - 2p\text{tr}(\tilde{G}_\gamma) + pd \\ &= \frac{p(2+3\theta) - \theta}{1+\theta} D[\tilde{G}_\gamma] : (\xi\xi^T) - p|\xi|^2 - 2p\text{tr}(\tilde{G}_\gamma) + pd \\ &= pd - p \sum_{i=1}^d \left[\xi_i^2 + 2\alpha_i - \alpha_p \alpha_i \xi_i^2 \right]. \end{aligned}$$

Note that $\underline{\alpha} \leq \alpha_i \leq \bar{\alpha}$, $\alpha_p \geq 2p$ by Remark 3.4 (i), and ξ_0^2 takes only values 0 and $\frac{1}{p}$. Then

$$\begin{aligned} \xi_i^2 + 2\alpha_i - \alpha_p \alpha_i \xi_i^2 &\leq (2\alpha_i) \vee \left[\frac{1}{p} + 2\alpha_i - \frac{\alpha_p}{p} \alpha_i \right] \leq 2\bar{\alpha} \vee \left[\frac{1}{p} + (2 - \frac{\alpha_p}{p}) \underline{\alpha} \right] \\ &= 2\bar{\alpha} = \frac{2\Lambda}{2p(\Lambda - 1) + \alpha_p}, \end{aligned}$$

thanks to Remark 3.4 (ii) and Assumption 3.3 (iii). Thus

$$I \geq pd - pd \frac{2\Lambda}{2p(\Lambda - 1) + \alpha_p}.$$

Note that the right side above is decreasing in Λ . Then, for h small enough,

$$1 - C\sqrt{h} + \frac{I}{1-p} \geq 1 + \frac{1}{1-p} \left[pd - pd \frac{2\lambda(p, \theta)}{2p(\lambda(p, \theta) - 1) + \alpha_p} \right] = 0,$$

where the last equality can be checked straightforwardly by using the definition of $\lambda(p, \theta)$. This, together with (3.12), proves the monotonicity. \square

Remark 3.6. In this remark we comment on the constraint $\Lambda < \Lambda_\theta$ and the optimal choice of σ_0 and p .

(i) We note that Λ and Λ_θ depend on σ_0 , but not on p . Moreover, they are invariant of the scaling of σ_0 . We shall first choose σ_0 among those with unit determinant to maximize $\frac{\Lambda_\theta}{\Lambda}$, and then rescale σ_0 as in Assumption 3.3 (iii).

(ii) When $\theta = 0$, namely \tilde{G}_γ is diagonal, we have $\Lambda_0 = \infty$. Then we remove the constraint (2.8) completely and thus improve the result of [15] significantly. In this case we may choose $p > 0$ small enough so that $\Lambda < \lambda(p, 0) = 1 + \frac{1}{p(d-1)}$. We also note that when $d = 1$ we always have $\theta = 0$.

(iii) When $(d-3)\theta < 2$ and $\theta \neq 0$, the optimal p is $\frac{\theta}{2-(d-3)\theta}$ which belongs to the interval $[\frac{\theta}{2(1+\theta)}, \frac{1}{3}]$. In this case $\Lambda_\theta = 1 + \frac{[2-(d-3)\theta]^2}{8\theta(1+\theta)(d-1)}$.

(iv) When $(d-3)\theta \geq 2$, the optimal p is $\frac{1}{3}$, and in this case $\Lambda_\theta = 1 + \frac{3(2-d\theta)}{2(1+\theta)(d-1)}$.

(v) When $\theta \geq \frac{2}{d}$, Assumption 3.3 is violated. In this case we may always set $p := \frac{1}{3}$ and our algorithm reduces back to [15], by replacing the Brownian motion there with trinomial tree. We may easily obtain the bounds (2.7) and (2.8) as in [15]. See also Remarks 3.1.

(vi) The above choices of p and the scaling of σ_0 is somewhat optimal in order to maintain the monotonicity. However, given G , they may not be optimal for the convergence of the scheme. In our numerical examples in Section 6 below, we may choose some different σ_0 and p . It is not clear to us how to choose p and σ_0 so as to optimize the efficiency of the algorithm.

Remark 3.7. This remark concerns the degeneracy of G .

(i) Unlike [15], we do not require $G_\gamma \geq \frac{1}{2}\sigma_0^2$ and thus F can be degenerate. This is possible mainly because we use bounded trinomial tree instead of unbounded Brownian motion.

(ii) Our condition $D[\tilde{G}_\gamma] > \underline{\alpha}I_d > \mathbf{0}$ is slightly weaker than the nondegeneracy requirement of [15]. In general, G_γ can be degenerate.

(iii) When $\underline{\alpha} = 0$, one can approximate the generator G by $G^\varepsilon := G + \varepsilon \sigma_0^2 : \gamma$ and numerically solve the corresponding solution u^ε . By the stability of viscosity solutions we see that u^ε converges to u locally uniformly.

(iv) By imposing some additional assumptions on G_z and by using the so called weak monotonicity, we may weaken Assumption 3.3 (ii) to $\Lambda \leq \Lambda_\theta$. We refer to [15] Assumption F (iii) for more details.

(v) Motivated from pricing Asian options, in a recent work Tan [27] investigated the numerical approximation for the following type of PDE with solution $u(t, x, y)$:

$$-\partial_t u(t, x, y) - G(t, x, y, u, D_x u, D_{xx}^2 u) - H(t, x, y, u, D_x u, D_y u) = 0,$$

where G is nondegenerate in $D_{xx}^2 u$, but the PDE is always degenerate in $D_{yy}^2 u$.

3.3 Stability

Given the monotonicity, one may prove stability following standard arguments.

Lemma 3.8. *Let Assumptions 2.1 and 3.3 hold. Then for any $h \in (0, h_0]$, \mathbb{T}_h^t satisfies the stability in Theorem 2.2 (iii).*

Proof. First, it follows from Lemma 3.5 that \mathbb{T}_h^t satisfies the monotonicity. Denote $C_n := \sup_{x \in \mathbb{R}^d} |u_h(t_n, x)|$ and $C_i := \sup_{(t, x) \in [t_i, t_{i+1}) \times \mathbb{R}^d} |u_h(t, x)|$, $i = n-1, \dots, 0$. Since g is bounded, we see that $C_n \leq C$. We claim that

$$C_i \leq (1 + Ch)C_{i+1} + Ch. \quad (3.13)$$

Then by the discrete Gronwall Inequality we see that

$$\max_{0 \leq i \leq n-1} C_i \leq C(1 + Ch)^n [C_n + nh] \leq C.$$

This proves the lemma.

We now prove (3.13). Let $t \in [t_i, t_{i+1})$ and denote $\tilde{h} := t_i - t \leq h$. Similar to (3.11), one may easily get

$$u_h(t, x) = \mathbb{E} \left[u_h(t_{i+1}, x + \sqrt{\tilde{h}} \sigma_0 \xi) I_{i+1} \right] + \tilde{h} F(t_i, x, 0, \mathbf{0}, \mathbf{0}) \quad (3.14)$$

where, for some deterministic $F_y(t_i), F_z(t_i), F_\gamma(t_i)$ defined in an obvious way,

$$I_{i+1} := 1 + \tilde{h} \left[F_y(t_i) + F_z(t_i) \cdot K_1(\xi) + F_\gamma(t_i) : K_2(\xi) \right].$$

The monotonicity in Lemma 3.5 exactly means $I_{i+1} \geq 0$. Then, noting that $F(t_i, x, 0, \mathbf{0}, \mathbf{0}) = G(t_i, x, 0, \mathbf{0}, \mathbf{0})$ is bounded,

$$|u_h(t, x)| \leq \mathbb{E} \left[|u_h(t_{i+1}, x + \sqrt{\tilde{h}} \sigma_0 \xi) | I_{i+1} \right] + C\tilde{h} \leq C_{i+1} \mathbb{E}[I_{i+1}] + C\tilde{h}.$$

By (3.6), we see that

$$E_{t_i}[I_{i+1}] = 1 + \tilde{h} F_y(t_i) \leq 1 + C\tilde{h}.$$

Then

$$|u_h(t, x)| \leq (1 + C\tilde{h})C_{i+1} + C\tilde{h} \leq (1 + Ch)C_{i+1} + Ch.$$

Since (t, x) is arbitrary, we obtain (3.13). \square

3.4 Boundary condition

It can be seen that the boundary condition holds if we can prove that $\lim_{(t', x', h) \rightarrow (T, x, 0)} u_h(t', x') = g(x)$ at T .

Lemma 3.9. *Let Assumptions 2.1 and 3.3 hold, then*

$$|u_h(t_k, x) - g(x)| \leq C(T - t_k)^{\frac{1}{2}}$$

Proof. Fix (t_k, x) . Let ξ^j , $j = k+1, \dots, n$ are independent d -dimensional random variables such that each ξ^j has independent components with distribution (3.2), and denote

$$X_{t_k}^n := x, \quad X_{t_j}^n := X_{t_{j-1}}^n + \sqrt{h} \sigma_0 \sum_{i=k+1}^j \xi^i, \quad j = k+1, \dots, n.$$

Then it is clear that, denoting $\mathcal{F}_j^n := \sigma(\xi^i, k+1 \leq i \leq j)$,

$$\begin{aligned} u_h(t_{j-1}, X_{t_{j-1}}^n) &= \mathbb{E}_{t_{j-1}}[u_h(t_j, X_{t_j}^n)] + hF(t_{j-1}, X_{t_{j-1}}^n, \mathbb{E}_{t_{j-1}}[u_h(t_j, X_{t_j}^n)], \\ &\quad \mathbb{E}_{t_{j-1}}[u_h(t_j, X_{t_j}^n)K_1(\xi^j)], \mathbb{E}_{t_{j-1}}[u_h(t_j, X_{t_j}^n)K_2(\xi^j)]). \end{aligned}$$

Similarly to the proof of Lemma 3.8, we have

$$u_h(t_{j-1}, X_{t_{j-1}}^n) = \mathbb{E}_{t_{j-1}}[u_h(t_j, X_{t_j}^n)I_j] + hF(t_{j-1}, X_{t_{j-1}}^n, 0, \mathbf{0}, \mathbf{0}),$$

where, by abusing the notation I slightly,

$$I_j := 1 + h \left[F_y(t_{j-1}) + F_z(t_{j-1}) \cdot K_1(\xi^j) + F_\gamma(t_{j-1}) : K_2(\xi^j) \right] \geq 0,$$

and $F_y(t_{j-1}), F_z(t_{j-1}), F_\gamma(t_{j-1})$ are defined in an obvious way. Denote

$$J_k := 1 \quad \text{and} \quad J_i := \Pi_{j=k+1}^i I_j, \quad i = k+1, \dots, n.$$

Recalling $u_h(t_n, x) = g(x)$, by induction we get

$$u_h(t_k, x) = u_h(t_k, X_{t_k}^n) = \mathbb{E} \left[g(X_{t_n}^n) J_n + h \sum_{j=k}^{n-1} J_j F(t_j, X_{t_j}^n, 0, \mathbf{0}, \mathbf{0}) \right].$$

Since g is bounded and uniformly Lipschitz continuous, we may let g_ϵ be a standard smooth molifier of g such that

$$\|g_\epsilon - g\|_\infty \leq C\epsilon, \quad \|Dg_\epsilon\|_\infty \leq C \quad \text{and} \quad \|D^2g_\epsilon\|_\infty \leq C\epsilon^{-1}. \quad (3.15)$$

Then, noting that $F(t, x, 0, \mathbf{0}, \mathbf{0})$ is bounded,

$$\begin{aligned} & |u_h(t_k, x) - g(x)| \\ & \leq \left| \mathbb{E}[g_\epsilon(X_{t_n}^n) J_n] - g_\epsilon(x) \right| + C\epsilon \mathbb{E}[J_n] + Ch \mathbb{E} \left[\sum_{j=k}^{n-1} J_j \right] + C\epsilon \\ & = \left| \sum_{i=k+1}^n \mathbb{E}[g_\epsilon(X_{t_i}^n) J_i - g_\epsilon(X_{t_{i-1}}^n) J_{i-1}] \right| + C\epsilon \mathbb{E}[J_n] + Ch \mathbb{E} \left[\sum_{j=k}^n J_j \right] + C\epsilon \\ & = \left| \sum_{i=k+1}^n \mathbb{E}[(g_\epsilon(X_{t_i}^n) - g_\epsilon(X_{t_{i-1}}^n)) J_i + g_\epsilon(X_{t_{i-1}}^n) J_{i-1} [I_i - 1]] \right| \\ & \quad + C\epsilon \mathbb{E}[J_n] + Ch \mathbb{E} \left[\sum_{j=k}^{n-1} J_j \right] + C\epsilon. \end{aligned}$$

Since $\mathbb{E}_{t_{j-1}}[I_j] = 1 + hF_y(t_{j-1})$, we have

$$\left| \mathbb{E}_{t_{i-1}}[I_i] - 1 \right| \leq Ch \quad \text{and} \quad \mathbb{E}[J_i] \leq (1 + Ch)^{i-k} \leq C. \quad (3.16)$$

Thus

$$\begin{aligned} & |u_h(t_k, x) - g(x)| \\ & \leq \left| \sum_{i=k+1}^n \mathbb{E}[(g_\epsilon(X_{t_i}^n) - g_\epsilon(X_{t_{i-1}}^n)) J_i] \right| + C(n-k)h + C\epsilon. \end{aligned} \quad (3.17)$$

Moreover, for some appropriate \mathcal{F}_{t_i} -measurable $\tilde{X}_{t_i}^n$,

$$g_\epsilon(X_{t_i}^n) - g_\epsilon(X_{t_{i-1}}^n) = \sqrt{h} Dg_\epsilon(X_{t_{i-1}}^n) \cdot \sigma_0 \xi^i + \frac{h}{2} D^2g_\epsilon(\tilde{X}_{t_i}^n) : (\sigma_0 \xi^i)(\sigma_0 \xi^i)^T.$$

Since ξ^i is bounded, we have

$$\begin{aligned} & \left| \mathbb{E}_{t_{i-1}} [Dg_\epsilon(X_{t_{i-1}}^n) \cdot \sigma_0 \xi^i I_i] \right| \\ &= \left| \mathbb{E}_{t_{i-1}} [h Dg_\epsilon(X_{t_{i-1}}^n) \cdot \sigma_0 \xi^i F_z(t_{i-1}) \cdot K_1(\xi^i)] \right| \leq C\sqrt{h}; \\ & \left| \mathbb{E}_{t_{i-1}} [D^2 g_\epsilon(\tilde{X}_{t_i}^n) : (\sigma_0 \xi^i)(\sigma_0 \xi^i)^T I_i] \right| \leq C\epsilon^{-1} \mathbb{E}_{t_{i-1}} [I_i] \leq C\epsilon^{-1}. \end{aligned}$$

Then

$$\left| \mathbb{E}_{t_{i-1}} [g_\epsilon(X_{t_i}^n) - g_\epsilon(X_{t_{i-1}}^n)] I_i \right| \leq Ch[1 + \epsilon^{-1}].$$

Plug this into (3.17) and recall (3.16), we have

$$|u_h(t_k, x) - g(x)| \leq C(n-k)h[1 + \epsilon^{-1}] + C\epsilon.$$

Note that $(n-k)h = T - t_k$. Set $\epsilon := \sqrt{T - t_k}$, we obtain the result. \square

3.5 Convergence Results

First, combine Lemmas 3.2, 3.5, 3.8, and 3.9, it follows immediately from Theorem 2.2 that

Theorem 3.10. *Let Assumptions 2.1 and 3.3 hold. Then u_h converges to u locally uniformly as $h \rightarrow 0$.*

We next study the rate of Convergence. We first consider the case that u is smooth. Let $C_b^{[4]}([0, T] \times \mathbb{R}^d)$ denote the set of bounded functions u such that the following derivatives exist and are continuous and bounded: $\partial_t u, D_x u, D_{xx}^2 u, \partial_{tt}^2 u, \partial_t D_x u, \partial_t D_{xx}^2 u, D_{xxx}^3 u, D_{xxxx}^4 u$.

Theorem 3.11. *Let Assumptions 2.1 and 3.3 hold and $h \in (0, h_0)$. Assume further that $u \in C_b^{[4]}([0, T] \times \mathbb{R}^d)$, and F is locally uniformly Lipschitz continuous in x , locally uniformly on (y, z, γ) . Then there exists a constant C , independent of h (or n), such that*

$$|u_h(t, x) - u(t, x)| \leq Ch \quad \text{for all } (t, x).$$

Proof. Again, since $h < h_0$, it follows from Lemma 3.5 that \mathbb{T}_h^t satisfies the monotonicity. Denote $C_n := \sup_{x \in \mathbb{R}^d} |u_h(t_n, x) - u(t_n, x)|$ and $C_i := \sup_{(t, x) \in [t_i, t_{i+1}] \times \mathbb{R}^d} |u_h(t, x) - u(t, x)|$, $i = n-1, \dots, 0$. We claim that

$$C_i \leq (1 + Ch)C_{i+1} + Ch^2. \quad (3.18)$$

Since $C_n = 0$, then by the discrete Gronwall Inequality we see that

$$\max_{0 \leq i \leq n-1} C_i \leq C(1 + Ch)^n [C_n + nh^2] \leq Ch.$$

This proves the theorem.

We now prove (3.18). Similar to the proof of Lemma 3.8, we shall only estimate $|u_h(t_i, x) - u(t_i, x)|$, and the estimate for the general $|u_h(t, x) - u(t, x)|$ is similar. For this purpose, recall (3.4), (3.5) and define

$$\begin{aligned} \tilde{u}_h(t_i, x) &:= [\mathcal{D}^0 u(t_{i+1}, \cdot)](x) \\ &\quad + hF\left(t_i, x, [\mathcal{D}^0 u(t_{i+1}, \cdot)](x), [\mathcal{D}^1 u(t_{i+1}, \cdot)](x), [\mathcal{D}^2 u(t_{i+1}, \cdot)](x)\right). \end{aligned} \quad (3.19)$$

We note that the right side of above uses the true solution u , instead of u_h in (2.3). It is clear that

$$|u_h(t_i, x) - u(t_i, x)| \leq |u_h(t_i, x) - \tilde{u}_h(t_i, x)| + |\tilde{u}_h(t_i, x) - u(t_i, x)|. \quad (3.20)$$

Compare (2.3) and (3.19), by the first equality of (3.12) we have

$$\begin{aligned} &u_h(t_i, x) - \tilde{u}_h(t_i, x) \\ &= \mathbb{E}\left[u_h - u(t_{i+1}, x + \sqrt{h}\sigma_0\xi) \left[1 + h[F_y + F_z \cdot K_1(\xi) + F_\gamma : K_2(\xi)]\right]\right] \end{aligned}$$

Then it follows from similar arguments in the proof of Lemma 3.8 that

$$|u_h(t_i, x) - \tilde{u}_h(t_i, x)| \leq (1 + Ch)C_{i+1}. \quad (3.21)$$

Next, since $u \in C_b^{[4]}([0, T] \times \mathbb{R}^d)$, applying Taylor expansion and by (3.3) we have

$$\begin{aligned} [\mathcal{D}^0 u(t_{i+1}, \cdot)](x) &= u(t_i, x) + \partial_t u(t_i, x)h + \frac{h}{2}D^2 u(t_i, x) : \sigma_0^2 + O(h^2); \\ [\mathcal{D}^1 u(t_{i+1}, \cdot)](x) &= Du(t_i, x) + O(h); \\ [\mathcal{D}^2 u(t_{i+1}, \cdot)](x) &= D^2 u(t_i, x) + O(h); \end{aligned}$$

Then

$$\begin{aligned} \tilde{u}_h(t_i, x) - u(t_i, x) &= \partial_t u(t_i, x)h + \frac{h}{2}D^2 u(t_i, x) : \sigma_0^2 + O(h^2) \\ &\quad + hF\left(\cdot, u + O(h), Du + O(h), D^2 u + O(h)\right)(t_i, x) \end{aligned}$$

Note that u satisfy the PDE (1.1) and recall (3.1), then

$$\begin{aligned} \left| \tilde{u}_h(t_i, x) - u(t_i, x) \right| &= O(h^2) + h \times \\ &\left| F(\cdot, u + O(h), Du + O(h), D^2u + O(h)) - F(\cdot, u, Du, D^2u) \right|(t_i, x). \end{aligned}$$

Since u and its derivatives are bounded, and F is locally uniformly Lipschitz continuous in x , then we have

$$\left| \tilde{u}_h(t_i, x) - u(t_i, x) \right| \leq Ch^2.$$

Plug this and (3.21) into (3.20), we obtain

$$\sup_x |u_h(t_i, x) - u(t_i, x)| \leq (1 + Ch)C_{i+1} + Ch^2.$$

Similarly we may estimate $\sup_x |u_h(t, x) - u(t, x)|$ for $t \in (t_i, t_{i+1})$, and thus prove (3.18). \square

We finally study the case when u is only a viscosity solution. Given the monotonicity, our arguments are almost identical to those of [15] Theorem 3.10, which in turn relies on the works Krylov [18] and Barles and Jakobsen [2]. We thus present only the result and omit the proof.

The result relies on the following additional assumption.

Assumption 3.12. (i) The generator G is of the Hamilton-Jacobi-Bellman type:

$$G(t, x, y, z, \gamma) = \inf_{\alpha \in \mathcal{A}} \left\{ \frac{1}{2} \sigma^\alpha (\sigma^\alpha)^T(t, x) : \gamma + b^\alpha(t, x)y + c^\alpha(t, x) \cdot z + f^\alpha(t, x) \right\},$$

where the functions σ^α , b^α , c^α and f^α are uniformly bounded, and uniformly Lipschitz continuous in x and uniformly Hölder- $\frac{1}{2}$ continuous in t , uniformly in α .

(ii) For any $\delta > 0$, there exists a finite set $\{\alpha_i\}_{i=1}^{M_\delta}$ such that for any $\alpha \in \mathcal{A}$:

$$\inf_{1 \leq i \leq M_\delta} (|\sigma^\alpha - \sigma^{\alpha_i}|_\infty + |b^\alpha - b^{\alpha_i}|_\infty + |c^\alpha - c^{\alpha_i}|_\infty + |f^\alpha - f^{\alpha_i}|_\infty) \leq \delta.$$

We then have the following result analogous to [15] Theorem 3.10.

Theorem 3.13. Let Assumptions 2.1 and 3.3 hold and $h \in (0, h_0)$.

(i) under Assumption 3.12 (i), we have $u - u_h \leq Ch^{1/4}$,

(ii) under the full Assumption 3.12, we have $-Ch^{1/10} \leq u - u_h \leq Ch^{1/4}$.

Remark 3.14. The arguments in [15] relies heavily on the viscosity properties of the PDE. Very recently Tan [26] provides a purely probabilistic arguments for HJB equations. His argument works for non-Markovian setting as well and thus provides a discretization for second order BSDEs. Moreover, under his conditions he shows that $|u_h - u| \leq Ch^{\frac{1}{8}}$, which improves the left side rate in Theorem 3.13 (ii).

4 Quasilinear PDE and Coupled FBSDEs

In this section we focus on following G which is quasilinear in γ :

$$G = \frac{1}{2}[\sigma\sigma^T](t, x, y) : \gamma + b(t, x, y, \sigma(t, x, y)z) \cdot z + f(t, x, y, \sigma(t, x, y)z). \quad (4.1)$$

Here f is scalar, b is \mathbb{R}^d -valued, and σ is $\mathbb{R}^{d \times m}$ -valued for some m . In this case the PDE (1.1) is closely related to the following coupled FBSDE:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t \sigma(s, X_s, Y_s)dW_s; \\ Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s \cdot dW_s. \end{cases} \quad (4.2)$$

Here W is a m -dimensional Brownian motion, and the solution triplet (X, Y, Z) takes values in \mathbb{R}^d , \mathbb{R} , and \mathbb{R}^m , respectively. Due to the four step scheme of Ma, Protter, and Yong [19], when the PDE (1.1) has the classical solution, the following nonlinear Feynman-Kac formula holds:

$$Y_t = u(t, X_t), \quad Z_t = \sigma(t, X_t, u(t, X_t))Du(t, X_t). \quad (4.3)$$

The feasible numerical method for high dimensional FBSDEs has been a challenging problem in the literature. There are very few papers on the subject, most of which are not feasible in high dimensional cases. To our best knowledge, the only work which reported a high dimensional numerical example is Bender and Zhang [7].

Our scheme works for quasilinear PDE as well, especially when $\sigma\sigma^T$ is diagonal (or close to diagonal in the sense that θ is small), and thus is appropriate for numerically solving FBSDE (4.2). We remark that the σ_0 we will choose is different from $\sigma(t, x, y)$ in (4.1), and the F defined by (3.1) is different from f . We shall present a 12-dimensional example, see Example 6.4 below.

One technical point is that the G in (4.1) is not Lipschitz continuous in y , mainly due to the term $\frac{1}{2}[\sigma\sigma^T](t, x, y) : \gamma$. This can be overcome when the PDE has a classical solution u . In general viscosity solution case, one may construct some approximating PDE which has classical solution and then use the stability of viscosity solutions, in the spirit of Remark 3.7 (iii).

Theorem 4.1. *Let G take the form (4.1). Assume*

(i) σ, b, f, g are bounded, continuous in all variables, and uniformly Lipschitz continuous in (x, y, z) ;

(ii) Assumption 3.3 holds;

(iii) The PDE (1.1) has a classical solution $u \in C_b^{[4]}([0, T] \times \mathbb{R}^d)$ (as introduced in Theorem 3.11).

Then $|u_h - u| \leq Ch$ when h is small enough.

Proof. We follow the proof of Theorem 3.11. Define C_i , $i = 0, \dots, n$ and \tilde{u}_h as in Theorem 3.11 and again it suffices to prove (3.18).

We first estimate $|u_h(t_i, x) - \tilde{u}_h(t_i, x)|$. Denote

$$\varphi_h(x) := u_h(t_{i+1}, x), \quad \varphi(x) := u(t_{i+1}, x), \quad \psi := \varphi_h - \varphi.$$

Then

$$\begin{aligned} & u_h(t_i, x) - \tilde{u}_h(t_i, x) \\ = & \mathcal{D}^0\psi(x) + hF\left(t_i, x, \mathcal{D}^0\varphi_h(x), \mathcal{D}^1\varphi_h(x), \mathcal{D}^2\varphi_h(x)\right) \\ & - hF\left(t_i, x, \mathcal{D}^0\varphi(x), \mathcal{D}^1\varphi(x), \mathcal{D}^2\varphi(x)\right) \\ = & \mathcal{D}^0\psi(x) - \frac{h}{2}\sigma_0^2 : \mathcal{D}^2\psi(x) + hG\left(t_i, x, \mathcal{D}^0\varphi_h(x), \mathcal{D}^1\varphi_h(x), \mathcal{D}^2\varphi_h(x)\right) \\ & - hG\left(t_i, x, \mathcal{D}^0\varphi(x), \mathcal{D}^1\varphi(x), \mathcal{D}^2\varphi(x)\right) \\ = & \mathcal{D}^0\psi(x) + h\left[\mathcal{L}^1\psi(x) + \mathcal{L}^2\psi(x) + \mathcal{L}^3\psi(x)\right], \end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}^1\psi(x) &:= -\frac{1}{2}\sigma_0^2 : \mathcal{D}^2\psi(x) + \frac{1}{2}[\sigma\sigma^T](t_i, x, \mathcal{D}^0\varphi_h(x)) : \mathcal{D}^2\psi(x) \\
&\quad + b(t_i, x, \mathcal{D}^0\varphi_h(x), \sigma(t_i, x, \mathcal{D}^0\varphi_h(x))\mathcal{D}^1\varphi_h(x)) \cdot \mathcal{D}^1\psi(x); \\
\mathcal{L}^2\psi(x) &:= \frac{1}{2}\left[[\sigma\sigma^T](t_i, x, \mathcal{D}^0\varphi_h(x)) - [\sigma\sigma^T](t_i, x, \mathcal{D}^0\varphi(x))\right] : \mathcal{D}^2\varphi(x) \\
\mathcal{L}^3\psi(x) &:= \left[b(t_i, x, \mathcal{D}^0\varphi_h(x), \sigma(t_i, x, \mathcal{D}^0\varphi_h(x))\mathcal{D}^1\varphi_h(x)) \right. \\
&\quad \left. - b(t_i, x, \mathcal{D}^0\varphi(x), \sigma(t_i, x, \mathcal{D}^0\varphi(x))\mathcal{D}^1\varphi(x))\right] \cdot \mathcal{D}^1\varphi(x) \\
&\quad + \left[f(t_i, x, \mathcal{D}^0\varphi_h(x), \sigma(t_i, x, \mathcal{D}^0\varphi_h(x))\mathcal{D}^1\varphi_h(x)) \right. \\
&\quad \left. - f(t_i, x, \mathcal{D}^0\varphi(x), \sigma(t_i, x, \mathcal{D}^0\varphi(x))\mathcal{D}^1\varphi(x))\right].
\end{aligned}$$

Let η denote a generic function with appropriate dimension which is uniformly bounded and may vary from line to line. Since u is smooth with bounded derivatives, one may easily check that $\mathcal{D}^0\varphi, \mathcal{D}^1\varphi, \mathcal{D}^2\varphi$ are bounded. Then

$$\begin{aligned}
\mathcal{L}^1\psi(x) &= \frac{1}{2}[\sigma\sigma^T](t_i, x, \mathcal{D}^0\varphi_h(x)) : \mathcal{D}^2\psi(x) - \frac{1}{2}\sigma_0^2 : \mathcal{D}^2\psi(x) + \eta \cdot \mathcal{D}^1\psi(x); \\
\mathcal{L}^2\psi(x) &= \eta\mathcal{D}^0\psi(x) \\
\mathcal{L}^3\psi(x) &= \eta\mathcal{D}^0\psi(x) \\
&\quad + \eta \cdot \left[\sigma(t_i, x, \mathcal{D}^0\varphi_h(x))\mathcal{D}^1\varphi_h(x) - \sigma(t_i, x, \mathcal{D}^0\varphi(x))\mathcal{D}^1\varphi(x)\right] \\
&= \eta\mathcal{D}^0\psi(x) + \eta \cdot \sigma(t_i, x, \mathcal{D}^0\varphi_h(x))\mathcal{D}^1\psi(x) \\
&\quad + \eta \cdot \left[\sigma(t_i, x, \mathcal{D}^0\varphi_h(x)) - \sigma(t_i, x, \mathcal{D}^0\varphi(x))\right]\mathcal{D}^1\varphi(x) \\
&= \eta\mathcal{D}^0\psi(x) + \eta \cdot \mathcal{D}^1\psi(x).
\end{aligned}$$

Then

$$\begin{aligned}
&u_h(t_i, x) - \tilde{u}_h(t_i, x) \\
&= \mathcal{D}^0\psi(x) + h\left[\eta\mathcal{D}^0\psi(x) + \eta \cdot \mathcal{D}^1\psi(x)\right] + \frac{h}{2}\left[\sigma\sigma^T - \sigma_0^2\right] : \mathcal{D}^2\psi(x) \\
&= \mathbb{E}\left[\psi(x + \sqrt{h}\sigma_0\xi)\left[1 + h\eta + h\eta \cdot K_1(\xi) + \frac{h}{2}[\sigma\sigma^T - \sigma_0^2] : K_2(\xi)\right]\right].
\end{aligned}$$

Now following the same arguments as in Lemma 3.5 we see that, for h small enough,

$$1 + h\eta + h\eta \cdot K_1(\xi) + \frac{h}{2}[\sigma\sigma^T - \sigma_0^2] : K_2(\xi) \geq 0.$$

Then it follows from the arguments in Theorem 3.11 that

$$|u_h(t_i, x) - \tilde{u}_h(t_i, x)| \leq (1 + Ch)C_{i+1}.$$

Similarly we may prove

$$|\tilde{u}_h(t_i, x) - u(t_i, x)| \leq Ch^2.$$

Thus we prove (3.18) and hence the theorem. \square

5 Monte Carlo Simulations

In this section we discuss how to implement the scheme. Fix $n \geq 1$ and $x \in \mathbb{R}^d$, our goal is to numerically compute $u_h(t_0, x)$. For this purpose, we need to apply the scheme (3.4)-(3.5) on trinomial trees. Let $\xi^i, i = 1, \dots, n$ be independent and each ξ^i takes values in \mathbb{R}^d and has independent components with distribution (3.2). Define the trinomial tree:

$$X_{t_i}^n := x + \sqrt{h} \sum_{j=1}^i \sigma_0 \xi^j, \quad i = 0, \dots, n. \quad (5.1)$$

We next define $Y_{t_n}^n := g(X_{t_n}^n)$, and for $i = n-1, \dots, 0$,

$$\begin{aligned} Y_{t_i}^n &:= \mathbb{E}_{t_i}[Y_{t_{i+1}}^n] \\ &+ hF\left(t_i, X_{t_i}^n, \mathbb{E}_{t_i}[Y_{t_{i+1}}^n], \mathbb{E}_{t_i}[Y_{t_{i+1}}^n K_1(\xi^{i+1})], \mathbb{E}_{t_i}[Y_{t_{i+1}}^n K_2(\xi^{i+1})]\right), \end{aligned} \quad (5.2)$$

where the kernels K_1 and K_2 are defined in (3.5). Then it is clear that

$$Y_{t_i}^n = u_h(t_i, X_{t_i}^n). \quad (5.3)$$

In particular, $u_h(t_0, x) = Y_{t_0}^n$. So it suffices to compute $Y_{t_0}^n$.

When the dimension is low, say $d \leq 3$, we may generate the whole trinomial tree, which consists of $\sum_{i=0}^n (2i+1)^d$ nodes. Then we may compute the exact value of Y^n at each node, where the conditional expectation in (5.2) is computed by the weighted average. This method is very efficient and the result is deterministic. It is in fact comparable to the standard finite difference method.

However, in above method the number of nodes grows exponentially in dimension d . When d is high, this method fails to work. As standard in the literature of BSDE numerics, in high dimensional case we use Monte Carlo approach to approximate the value of $Y_{t_0}^n$. In particular, we shall use

the least square regression method as introduced in Gobet, Lemor and Waxin [17]. We remark that Monte Carlo method is less sensitive to dimensions. For example, it can be seen in next section that we can use Monte Carlo method to approximate a 12-dimensional PDE with 160 time steps and 13333333 paths, while for finite difference method 13333333 paths is insufficient for 2 time steps when $d = 12$.

Roughly speaking, for each t_i , fix appropriate basis functions $\phi_j(t_i, \cdot)$, $j = 1, \dots, J$. We approximate conditional expectations $\mathbb{E}_{t_i}[Y_{t_{i+1}}^n K_k(\xi^{i+1})]$, $k = 0, 1, 2$, in (5.2) by their least square regression on the linear span of $\{\phi_j(t_i, X_{t_i}^n)\}_{1 \leq j \leq J}$, that is, by $\sum_{j=1}^J \alpha_j \phi_j(t_i, X_{t_i}^n)$ where $\{\alpha_j\}_{1 \leq j \leq J}$ are $\sigma(X_{t_i}^n)$ measurable random variables minimizing

$$\mathbb{E} \left[\left| \sum_{j=1}^J \alpha_j \phi_j(t_i, X_{t_i}^n) - Y_{t_{i+1}}^n K_k(\xi^{i+1}) \right|^2 \middle| X_{t_i} \right].$$

We next simulate L -paths of X^n and use them to approximate the coefficients α_j . We refer to [17] for the details as well as the error analysis.

We note that there are three types of errors in this algorithm: the time discretization error, the least square regression error, and the Monte Carlo simulation error. The first error is already analyzed in Sections 3 and 4. The third error is small when L is large, due to the Central Limit Theorem. The second error relies heavily on our choices of the basis functions. While there are some studies on how to choose good basis functions, see e.g. Bender and Steiner [6], overall speaking this is still an open problem. Usually g and its derivatives (when exist) are good candidates of basis functions.

In our numerical examples in next section, we shall focus our attention to the discretization error and the simulation error. That is, for the examples where we know the true solution, we will choose basis functions whose linear span contains the true solution, and thus the regression error vanishes. We shall leave the choice of basis functions for future study.

6 Numerical Examples

In this section we apply our scheme to various examples.

6.1 Examples under Monotonicity Condition

In this subsection we consider examples with diagonal G_γ and we shall always choose σ_0 diagonal, so $\theta = 0$ and thus $\Lambda_\theta = \infty$ in Assumption 3.3. Therefore, we have no constraint on Λ .

We start with a 3-dimensional example for which we can compute its values over the trinomial tree by using the weighted averages.

Example 6.1. A 3-dimensional Fully Nonlinear PDE

$$\begin{cases} -\partial_t u - \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} [(\sigma^2 I_d) : D^2 u] + f(t, x, u, Du) = 0, & \text{on } [0, T) \times \mathbb{R}^d, \\ u(T, x) = \sin(T + x_1 + \dots + x_d), & \text{on } \mathbb{R}^d, \end{cases} \quad (6.1)$$

where $0 < \underline{\sigma} < \bar{\sigma}$ are both in \mathbb{R} , and

$$f(t, x, y, z) = \frac{1}{d} \sum_{i=1}^d z_i - \frac{d}{2} \inf_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} (\sigma^2 y). \quad (6.2)$$

It can be checked that (6.1) has a classical solution:

$$u(t, x) = \sin(t + x_1 + \dots + x_d), \quad (6.3)$$

with which we can verify the convergence of our numerical approximation.

To test its convergence under different nonlinearities, we assume that $d = 3$, $\underline{\sigma} = 1$, $\bar{\sigma} = \sqrt{2}$, $\sqrt{4}$, or $\sqrt{6}$. Supposing that $T = 0.5$ and $x_0 = (5, 6, 7)$, we know the true solution is $u(0, x_0) = \sin(5 + 6 + 7) \approx -0.750987$.

According to our scheme, when \tilde{G}_γ is diagonal, $\theta = 0$, which implies that we can choose the following parameters: $\Lambda = \frac{\bar{\sigma}^2}{\underline{\sigma}^2}$, $p = \min \left\{ \frac{1}{(\Lambda-1)(d-1)}, \frac{1}{3} \right\}$, $\alpha_p = 2$, $\underline{\alpha} = \frac{1}{2p(\Lambda-1)+\alpha_p}$, and $\sigma_0 = \frac{\underline{\sigma}}{\sqrt{2\underline{\alpha}}} I_d$. We remark that $\Lambda = 2, 4, 6$ respectively, which violates the constraint (2.8) of [15]. Denote the number of time partitions by n . By applying the weighted average method we can obtain the results in Figure 1, where the cost in time increases from 0.1 second to 800 seconds exponentially as n increases from 20 to 160 linearly. The table in Figure 1 contains the numerical solutions when $\bar{\sigma}^2 = 2$ exclusively, while the graph depicts the errors under three different choices of $\bar{\sigma}^2$.

As we can see from Figure 1, the rate of convergence is approximately $C \cdot h$, whereas the C depends on the structure of G . Therefore, our scheme works generally for large Λ when \tilde{G} is diagonal or diagonally dominant with a small θ .

In Figure 2 we compare the convergence of our scheme with that of finite difference method by fixing $\underline{\sigma} = 1$, $\bar{\sigma} = \sqrt{2}$. It can be seen that our result converges slightly slower than, but is comparable to, the finite difference method in solving low dimensional problems.

To see more of our scheme in extreme condition, we assume $\underline{\sigma} = 0$. Then we truncate G_γ from below with a positive definite matrix $\varepsilon I_d > 0$. That is,

n	Approx. $\bar{\sigma}^2 = 2$
20	-0.72984
40	-0.74028
60	-0.74382
80	-0.74667
100	-0.74560
120	-0.74738
140	-0.74790
160	-0.74829
Ans.	-0.750987

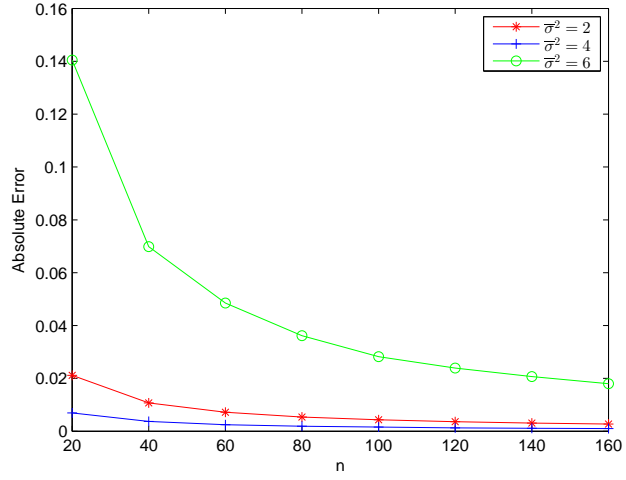


Figure 1: A 3-dimensional example with various degrees of non-linearity in Example 6.1.

n	Ours	F.D.
20	-0.72984	-0.76420
40	-0.74028	-0.75785
60	-0.74382	-0.75562
80	-0.74667	-0.75447
100	-0.74560	-0.75379
120	-0.74738	-0.75332
140	-0.74790	-0.75300
160	-0.74829	-0.75274
Ans	-0.75099	-0.75099

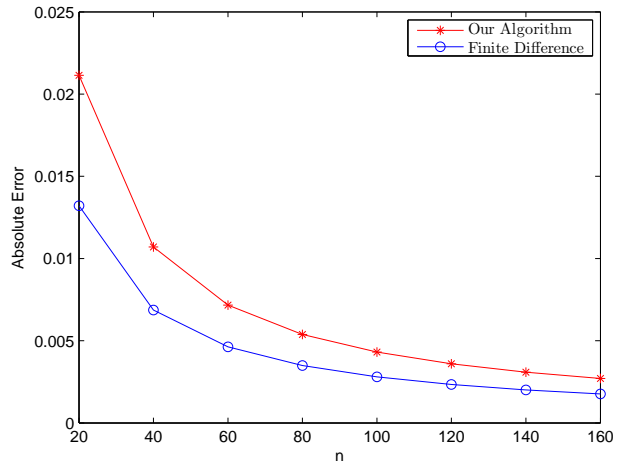


Figure 2: Comparison with finite difference method in Example 6.1.

n	Approx. $\bar{\sigma}^2 = 2$
20	-0.76285
40	-0.75705
60	-0.75508
80	-0.75401
100	-0.75339
120	-0.75297
140	-0.75269
160	-0.75247
Ans.	-0.750987

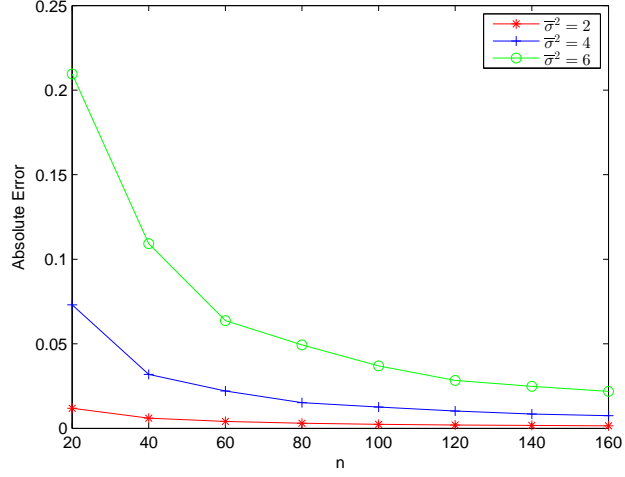


Figure 3: Convergence of a degenerate PDE truncated in Example 6.1.

we approximate (6.1) by the following nondegenerate PDE:

$$\begin{aligned}
& -\partial_t u - \frac{1}{2} \sup_{\varepsilon \leq \sigma \leq \bar{\sigma}} \left[(\sigma^2 I_d) : D^2 u \right] + f(t, x, u, Du) = 0, \quad 0 \leq t \leq T, \\
& u(T, x) = \sin(T + x_1 + \dots + x_d), \quad \text{on } \mathbb{R}^d, \quad \varepsilon = 0.01,
\end{aligned}$$

where f is given by (6.2). Figure 3 shows the feasibility of truncation in dealing with $\underline{\sigma} = 0$. \square

Our main motivation is to provide an efficient algorithm for high dimensional PDEs. At below we test our scheme on a twelve dimensional example, for which we shall use the the regression-based Monte Carlo method suggested in [17].

Example 6.2. A 12-dimensional example

Consider the PDE (6.1) with $d = 12$, $\underline{\sigma} = 1$, $\bar{\sigma} = \sqrt{2}$,

$$f(t, x, y, z) = \cos\left(t + \sum_{i=1}^d x_i\right) - \frac{d}{2} \inf_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} \left(\sigma^2 \sin\left(t + \sum_{i=1}^d x_i\right) \right). \quad (6.4)$$

The true solution is $u(t, x) = \sin(t + \sum_{i=1}^d x_i)$ again. As explained in Section 5, in this paper we want to focus on the discretization error and simulation error, so we rule out the regression error and test our algorithm by using the following perfect set of basis functions:

$$1, \quad x, \quad \sin(T + x_1 + \dots + x_d), \quad \cos(T + x_1 + \dots + x_d).$$

n	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
2	2083	160	0.659639	3.53×10^{-6}	4.48×10^{-2}
5	13021	64	0.562635	1.99×10^{-6}	1.46×10^{-1}
10	52083	32	0.546598	8.41×10^{-7}	1.17×10^0
20	208333	16	0.530432	8.04×10^{-7}	1.08×10^1
40	833333	8	0.521343	2.25×10^{-7}	9.11×10^1
80	3333333	4	0.519701	1.21×10^{-7}	7.28×10^2
160	13333333	2	0.517363	6.17×10^{-8}	5.86×10^3
True Solution			0.513978		

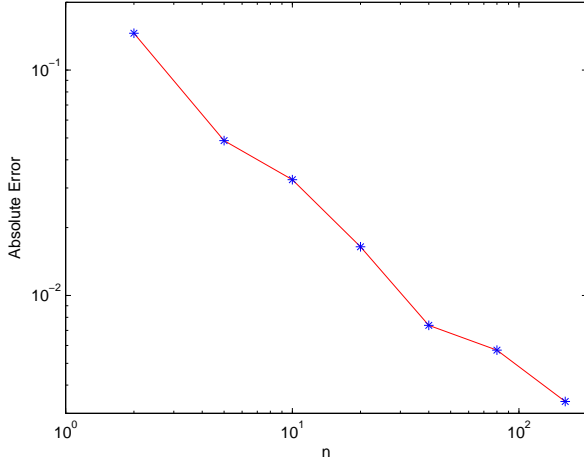


Figure 4: Numerical results of a 12-dimensional example in Example 6.2.

To test the result, we fix $T = 0.2$ and $x_0 = \{1, 2, \dots, 12\}$, which implies that the true solution is $\sin(78) = 0.513978$. Assuming that we repeat K identical and independent tests, and we sample L paths in each test. Denoting the average of the results in K tests by $\text{Avg}(\text{Ans.})$, and the variance of this average by $\text{Var}(\text{Avg.})$, we can obtain the results in the table and figure in Figure 4, where we conduct fewer tests for larger L , because the results are stable enough to draw our conclusion.

It can be seen from Figure 4 that the error shrinks slightly slower than $O(h)$, which is due to the simulation error. Hence we want to explore the influence of simulation error by using all the parameters as above but fixing $n = 40$, $K = 2$, $d = 12$, $T = 0.2$, $n = 40$, $\underline{\sigma} = 1$, $\bar{\sigma} = \sqrt{2}$. we increase L , the number of paths sampled, to see how the error reduces in Figure 5. While the variance and error decrease with more paths sampled, the cost in time increases linearly with respect to L from 8 seconds to 1400 seconds in Figure 5.

L	Approx.
83333	0.543643
166667	0.526979
416667	0.523897
833333	0.524683
1666667	0.521531
3333333	0.521017
6666667	0.520083
13333333	0.518607
Ans.:	0.513978

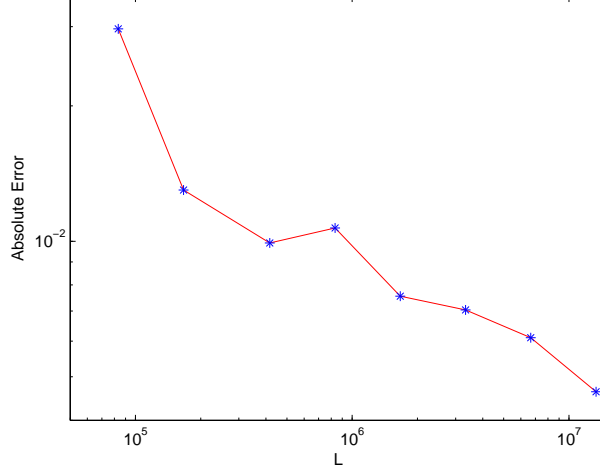


Figure 5: Relation between size of sample and errors in Example 6.2.

We have seen that our scheme converges to the true classical solution if it exists. Meanwhile, if the PDE only has a unique viscosity solution, our scheme can render a converging result as well.

Let f be zero in the Equation (6.1), then this equation has some unknown viscosity solution. However, our numerical results in Figure 6 still demonstrate a converging sequence. This can be also be observed from the decreasing differences between the numerical results. We shall remark though in this case our choice of basis functions may not be good. Again, we leave the analysis of the basis functions to future study. \square

It is well known that Isaacs equations have a unique viscosity solution under mild technical conditions. We next test our scheme on the following Isaacs equation to see its performance.

Example 6.3. A 12-dimensional Isaacs equation with unknown viscosity solution

$$\begin{cases} -u_t - G(D^2u) = 0, & \text{on } [0, T) \times \mathbb{R}^d, \\ u(T, \cdot) = \sin(T + x_1 + \dots + x_d), & \text{on } \mathbb{R}^d, \end{cases}$$

where

$$\begin{aligned} G(\gamma) &:= \sum_{i=1}^d \sup_{0 \leq u \leq 1} \inf_{0 \leq v \leq 1} \left[\frac{1}{2} \sigma^2(u, v) \gamma_{ii} + f(u, v) \right] \\ &= \sum_{i=1}^d \inf_{0 \leq v \leq 1} \sup_{0 \leq u \leq 1} \left[\frac{1}{2} \sigma^2(u, v) \gamma_{ii} + f(u, v) \right], \end{aligned}$$

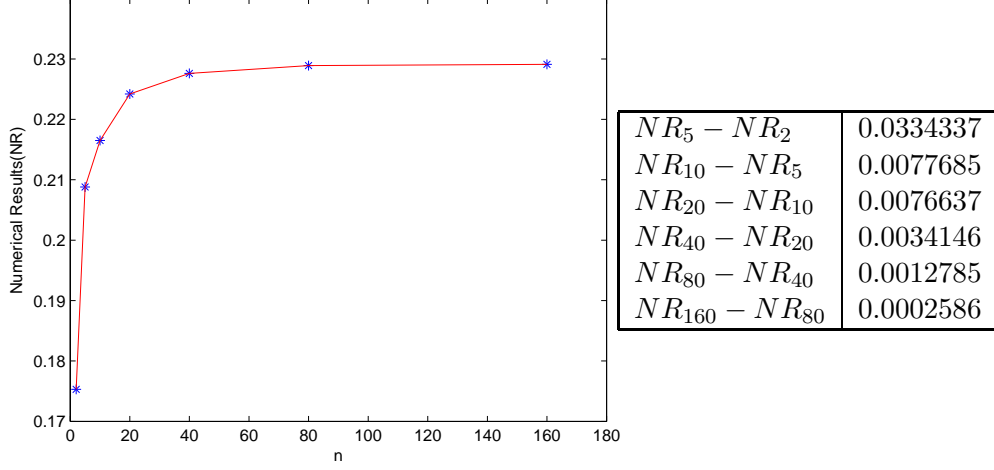


Figure 6: Numerical results for a PDE with unknown viscosity solution in Example 6.2.

and

$$\sigma^2(u, v) := (1 + u + v), \quad f(u, v) := -\frac{u^2}{4} + \frac{v^2}{4}.$$

One can easily simplify $G(\gamma)$ as: $G(\gamma) = \sum_{i=1}^d g(\gamma_{ii})$ where

$$g(\gamma_{ii}) := \begin{cases} \gamma_{ii} - \frac{1}{4}, & \gamma_{ii} \in (1, +\infty), \\ \frac{\gamma_{ii}}{2} + \frac{(\gamma_{ii}^+)^2}{4} - \frac{(\gamma_{ii}^-)^2}{4}, & \gamma_{ii} \in [-1, 1], \\ \gamma_{ii} + \frac{1}{4}, & \gamma_{ii} \in (-\infty, -1). \end{cases}$$

Therefore $G(\gamma)$ is neither concave nor convex when $\gamma = 0$. Setting $d=12$, we assign arbitrary initial value $x_0 = \{x_0^{(i)}\}_{i=1}^d$ to inspect the outcome. One example tested here is $x_0^{(i)} = 2i\pi - \frac{T-0.5\pi}{d}$.

Though the viscosity solution is unknown, our scheme still renders a converging numerical result in Figure 7. \square

We next test our scheme for a 12-dimensional coupled FBSDE.

Example 6.4. A 12-dimensional coupled FBSDE

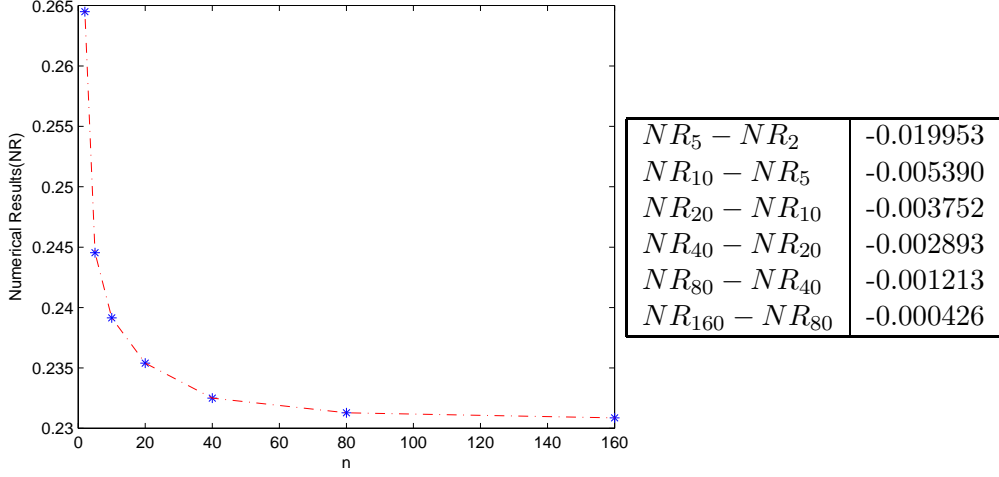


Figure 7: A 12-dimensional Isaacs equation with unknown viscosity solution in Example 6.3.

Consider FBSDE (4.2) with $m = d = 12$, σ is diagonal, and

$$\begin{aligned}
 b_i(t, x, y, z) &:= \cos(y + z_i), \quad \sigma_{ii}(t, x, y) := 1 + \frac{1}{3} \sin\left(\frac{1}{d} \sum_{j=1}^d x_j + y\right), \\
 f(t, x, y, z) &:= \frac{d}{2} \sin\left(t + \sum_{i=1}^d x_i\right) \left[1 + \frac{1}{3} \sin\left(\frac{1}{d} \sum_{i=1}^d x_i + y\right)\right]^2 \\
 &\quad - \frac{\frac{1}{d} \sum_{i=1}^d z_i}{1 + \frac{1}{3} \sin\left(\frac{1}{d} \sum_{j=1}^d x_j + y\right)} - d \cos\left(t + \sum_{i=1}^d x_i\right) \cos\left(y + \cos\left(t + \sum_{i=1}^d x_i\right)\right); \\
 g(x) &:= \sin\left(T + \sum_{i=1}^d x_i\right).
 \end{aligned}$$

The associated PDE (4.1) looks quite complicated, however, the coefficients are constructed in a way so that $u(t, x) := \sin(t + \sum_{i=1}^d x_i)$ is the classical solution. Consequently, denoting $\bar{X}_t := \frac{1}{d} \sum_{j=1}^d X_t^j$,

$$Y_t = \sin(t + d\bar{X}_t), \quad Z_t^i = \cos(t + d\bar{X}_t) \left[1 + \frac{1}{3} \sin(\bar{X}_t + \sin(t + d\bar{X}_t))\right]$$

satisfy the FBSDE.

For PDE (4.1), we see that $G_\gamma = \frac{1}{2}\sigma^2$ is diagonal and $\frac{2}{3} \leq \sigma_{ii} \leq \frac{4}{3}$ for each i . We note that f is not bounded and not Lipschitz continuous in y , however, since Z is bounded, then $f(t, x, y, Z_t)$ is bounded and Lipschitz

n	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
2	2083	160	1.462543	3.35×10^{-5}	1.56×10^{-2}
5	13021	64	1.111675	2.30×10^{-5}	2.36×10^{-1}
10	52083	32	1.014295	2.48×10^{-5}	2.43×10^0
20	208333	16	0.925712	8.10×10^{-6}	2.29×10^1
40	833333	8	0.912373	2.46×10^{-6}	1.94×10^2
80	3333333	4	0.908013	2.89×10^{-7}	1.56×10^3
160	13333333	2	0.888747	1.62×10^{-8}	3.42×10^4

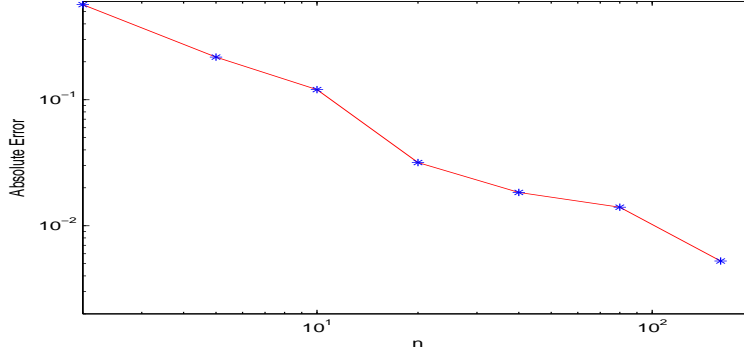


Figure 8: A 12-dimensional coupled FBSDE in Example 6.4.

continuous in y , and thus actually we may still apply Theorem 4.1. Set $d = 12$, $T = 0.2$, $X_0 = (2, 3, 4, \dots, 13)$, and apply the parameters specified before for our scheme. An approximation of Y_0 is shown in Figure 8, where the true solution $Y_t = \sin(t + \sum_{i=1}^d X_t^i)$ has value 0.893997 at $t=0$. \square

6.2 Examples violating Monotonicity Condition

In this subsection we apply our scheme to some examples which do not satisfy our monotonicity Assumption 3.3. So theoretically we do not know if our scheme converges or not. However, our numerical results show that the approximation still converges to the true solution. It will be very interesting to understand the scheme under these situations, and we shall leave it for future research.

Example 6.5. A 12-dimensional PDE with $\underline{\sigma} = 0$

Consider the same setting as Example 6.2 except that $\underline{\sigma} = 0$.

Instead of truncating G_γ as we did at the end of Example 6.1, we will pick parameters p and σ_0 as if $\underline{\sigma}$ were some small positive number. Then Assumption 3.3 is violated and our scheme is in fact not monotone. Nevertheless, our numerical results show that our approximations still converge to the true solution, as presented in Figure 9.

n	Approx.
2	0.22363
5	0.28971
10	0.38098
20	0.44215
40	0.47712
80	0.49699
160	0.50097
Ans.	0.51398

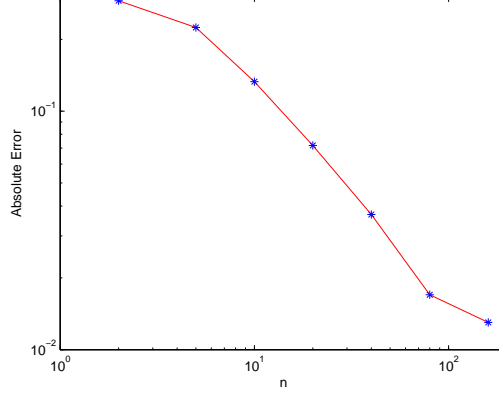


Figure 9: A 12-dimensional example without monotonicity in Example 6.5. □

We next apply our scheme to the following HJB equation which is associated with a Markovian second order BSDEs, introduced by [9] and [25]:

$$\begin{cases} -\frac{\partial u}{\partial t} - \frac{1}{2} \sup_{\underline{\sigma} \leq \sigma \leq \bar{\sigma}} [\sigma^2 : D^2 u] - f(t, x) = 0, & \text{on } [0, T) \times \mathbb{R}^d, \\ u(T, x) = g(x), & \text{on } \mathbb{R}^d, \end{cases} \quad (6.5)$$

When $f = 0$, this PDE induces exactly the G -expectation introduced by Peng [24]. We emphasize that, unlike in previous examples, here $\underline{\sigma}, \bar{\sigma}, \sigma \in \mathbb{S}^d$ are matrices and $\mathbf{0} < \underline{\sigma} \leq \sigma \leq \bar{\sigma}$. In particular, G_γ is not diagonal anymore. We remark that one has a representation for the solution of this PDE in terms of stochastic control:

$$u(0, x) = \sup_{\sigma} \mathbb{E} \left[g(X_T^\sigma) + \int_0^T f(t, X_t^\sigma) dt \right], \quad X_t^\sigma := x + \int_0^t \sigma_s dW_s,$$

where W is a d -dimensional Brownian motion, and the control σ is an \mathbb{F}^W -progressively measurable \mathbb{S}^d -valued process such that $\underline{\sigma} \leq \sigma \leq \bar{\sigma}$. Due to this connection, these kind of PDEs and the related G -expectation and second order BSDEs are important in applications with diffusion control and/or volatility uncertainty.

Example 6.6. A 10-dimensional HJB equation

Consider the PDE (6.5) with $g(x) = \sin(T + x_1 + \frac{x_2}{2} \dots + \frac{x_d}{d})$ and appropriate $f(t, x)$ so that

$$u(t, x) = \sin(t + x_1 + \frac{x_2}{2} + \dots + \frac{x_d}{d})$$

is the true solution to the PDE. We set $d = 10$.

To begin our test, we select randomly an initial point X_0 , and two 10-dimensional positive definite matrices $\bar{\sigma}^2$ and $\underline{\sigma}^2$. Then we obtain the numerical results displayed in Figure 10, which shows that our method works well for high dimensional HJB equation. The parameters used in this example are:

$$X_0 = (2.99, 3.05, 1.54, 1.89, 2.52, 1.10, 3.21, 1.64, 1.02, 1.80),$$

which gives a true solution 0.75805,

$$\bar{\sigma}^2 = \begin{pmatrix} 1.18, -0.35, -0.29, 0.23, -0.52, 0.09, -0.09, 0.21, 0.25, -0.03 \\ -0.35, 2.84, 0.42, -0.23, 0.00, -0.03, 0.21, -0.38, -0.25, 0.73 \\ -0.29, 0.42, 1.54, -1.17, 0.32, -0.16, -0.64, -0.63, -0.35, -0.12 \\ 0.23, -0.23, -1.17, 2.54, -0.30, 0.07, 0.30, 0.97, 0.43, 0.22 \\ -0.52, 0.00, 0.32, -0.30, 1.77, 0.25, 0.09, -0.39, 0.19, 0.13 \\ 0.09, -0.03, -0.16, 0.07, 0.25, 2.13, 0.23, 0.82, 0.65, 0.42 \\ -0.09, 0.21, -0.64, 0.30, 0.09, 0.23, 1.79, 0.31, 0.06, -0.30 \\ 0.21, -0.38, -0.63, 0.97, -0.39, 0.82, 0.31, 1.93, 0.14, 0.88 \\ 0.25, -0.25, -0.35, 0.43, 0.19, 0.65, 0.06, 0.14, 1.39, -0.05 \\ -0.03, 0.73, -0.12, 0.22, 0.13, 0.42, -0.30, 0.88, -0.05, 1.76 \end{pmatrix},$$

and

$$\underline{\sigma}^2 = \begin{pmatrix} 0.73, -0.21, -0.09, 0.28, -0.18, -0.07, -0.07, -0.16, 0.20, -0.22 \\ -0.21, 1.63, 0.15, -0.04, -0.07, -0.30, -0.04, -0.40, -0.19, 0.08 \\ -0.09, 0.15, 1.06, -0.80, 0.18, -0.28, -0.64, -0.66, -0.35, -0.06 \\ 0.28, -0.04, -0.80, 1.31, -0.07, 0.57, 0.19, 0.60, 0.69, 0.15 \\ -0.18, -0.07, 0.18, -0.07, 0.38, 0.10, -0.23, -0.04, -0.12, 0.01 \\ -0.07, -0.30, -0.28, 0.57, 0.10, 0.54, -0.16, 0.53, 0.18, 0.29 \\ -0.07, -0.04, -0.64, 0.19, -0.23, -0.16, 1.32, 0.04, 0.06, -0.53 \\ -0.16, -0.40, -0.66, 0.60, -0.04, 0.53, 0.04, 0.98, 0.17, 0.61 \\ 0.20, -0.19, -0.35, 0.69, -0.12, 0.18, 0.06, 0.17, 0.81, -0.11 \\ -0.22, 0.08, -0.06, 0.15, 0.01, 0.29, -0.53, 0.61, -0.11, 1.09 \end{pmatrix}$$

One can check that $\bar{\sigma}^2 > \underline{\sigma}^2$ because the smallest eigenvalue of $\bar{\sigma}^2 - \underline{\sigma}^2$ is 0.0026, which is positive. \square

Note that the PDE (6.5) involves the computation of $\sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} [\sigma^2 : \gamma]$. We provide some discussion below.

Remark 6.7. Let $\bar{\sigma}^2 - \underline{\sigma}^2 = LL^T$ be the Cholesky Decomposition, namely L is a $d \times d$ lower triangular matrix. Then for any $\gamma \in \mathbb{S}^d$, we have

$$\sup_{\underline{\sigma}^2 \leq \sigma^2 \leq \bar{\sigma}^2} [\sigma^2 : \gamma] = \underline{\sigma}^2 : \gamma + \sum_{i=1}^n \hat{\gamma}_i^+,$$

where $\hat{\gamma}_i$, $i = 1, \dots, d$, are the eigenvalues of $L^T \gamma L$.

n	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
5	10000	40	0.78385	5.22×10^{-8}	13
10	10000	20	0.77542	4.28×10^{-7}	57
15	10000	13	0.77202	3.80×10^{-7}	135
20	10000	10	0.76997	4.45×10^{-7}	248
25	10000	8	0.76930	2.28×10^{-6}	395
30	10000	6	0.76696	3.25×10^{-6}	573
35	10000	5	0.76683	3.08×10^{-6}	784

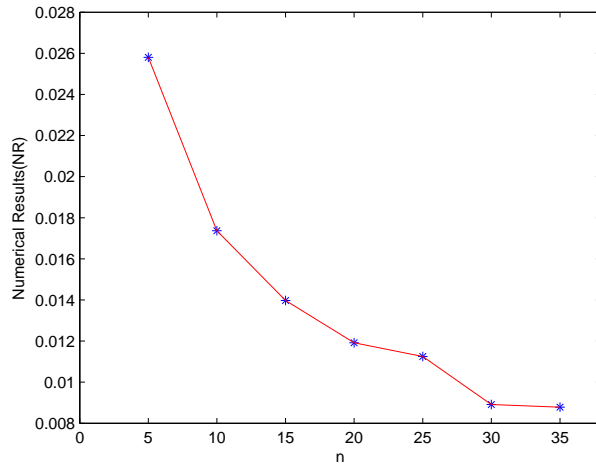


Figure 10: A 10-dimensional HJB equation in Example 6.6.

Proof. Obviously, any $\sigma^2 \in \mathbb{S}^d$ between $\underline{\sigma}^2$ and $\bar{\sigma}^2$ can be expressed as $\sigma^2 = \underline{\sigma}^2 + A$, where $\mathbf{0} \leq A \leq LL^T$. Then $\mathbf{0} \leq L^{-1}AL^{-T} \leq I_d$. We make the following eigenvalue decompositions:

$$L^{-1}AL^{-T} = U\hat{A}U^T, \quad L^T\gamma L = P\hat{\gamma}P^T,$$

where $UU^T = PP^T = I_d$, and \hat{A} and $\hat{\gamma}$ diagonal matrices. It is clear that the diagonal terms of \hat{A} are $\hat{a}_i \in [0, 1]$ and the diagonal terms of $\hat{\gamma}$ are $\hat{\gamma}_i$. Denote $Q := U^TP$. Then

$$\begin{aligned} \sigma^2 : \gamma - \underline{\sigma}^2 : \gamma &= A : \gamma = [L^{-1}AL^{-T}] : [L^T\gamma L] = [U\hat{A}U^T] : [L^T\gamma L] \\ &= \hat{A} : [U^TL^T\gamma LU] = \hat{A} : [Q\hat{\gamma}Q^T] \\ &= \sum_{i=1}^d \hat{a}_i \sum_{j=1}^d q_{ij}^2 \hat{\gamma}_j \leq \sum_{i=1}^d \left(\sum_{j=1}^d q_{ij}^2 \hat{\gamma}_j \right)^+ \end{aligned}$$

Note that $\sum_{j=1}^d q_{ij}^2 = 1$. Then by Jensen's inequality,

$$\sigma^2 : \gamma - \underline{\sigma}^2 : \gamma \leq \sum_{i=1}^d \left(\sum_{j=1}^d q_{ij}^2 \hat{\gamma}_j \right)^+ \leq \sum_{i=1}^d \sum_{j=1}^d q_{ij}^2 \hat{\gamma}_j^+ = \sum_{j=1}^d \hat{\gamma}_j^+ \sum_{i=1}^d q_{ij}^2 = \sum_{j=1}^d \hat{\gamma}_j^+.$$

This proves the remark.

Moreover, from the proof we see that the equality holds when

$$\hat{a}_i = 1_{\{\sum_{j=1}^d q_{ij}^2 \hat{\gamma}_j > 0\}} \quad \text{and} \quad Q = I_d.$$

That is, $U = P$ and thus $\sigma^2 = \underline{\sigma}^2 + LP\hat{A}P^TL^T$, where \hat{A} is the diagonal matrix whose diagonal terms are $\hat{a}_i = 1_{\{\hat{\gamma}_i > 0\}}$. \square

We remark that the above computation is in fact quite time consuming. At below we provide another example where G_γ is tridiagonal and the scheme is much more efficient.

Example 6.8. A 10-dimensional example with tridiagonal structure

Consider the PDE (1.1) with

$$\begin{aligned} G(t, x, y, z, \gamma) &:= \left(3 \sum_{i=1}^d \gamma_{ii} + \sum_{|i-j|=1} \frac{1}{\sqrt{1+(\gamma_{ij})^2}} \right) + f(t, x) \\ g(x) &:= \sin(T + x_1 + x_2 + \dots + x_d), \end{aligned} \quad (6.6)$$

and $f(t, x)$ is a chosen function such that $u(t, x) := \sin(t + x_1 + \dots + x_d)$ is the true solution of the PDE.

n	L	K	Avg(Ans.)	Var(Avg.)	cost (in seconds)
2	2500	160	-1.47362	1.0×10^{-5}	1.2×10^{-2}
5	15625	64	-1.15004	1.7×10^{-6}	1.4×10^{-1}
10	62500	32	-1.06194	9.1×10^{-6}	1.0×10^0
20	250000	16	-1.04519	2.1×10^{-6}	8.9×10^0
40	1000000	8	-1.03326	6.2×10^{-7}	7.1×10^1
80	4000000	4	-1.03092	5.8×10^{-8}	5.9×10^2
160	16000000	2	-1.01910	3.0×10^{-9}	1.4×10^4

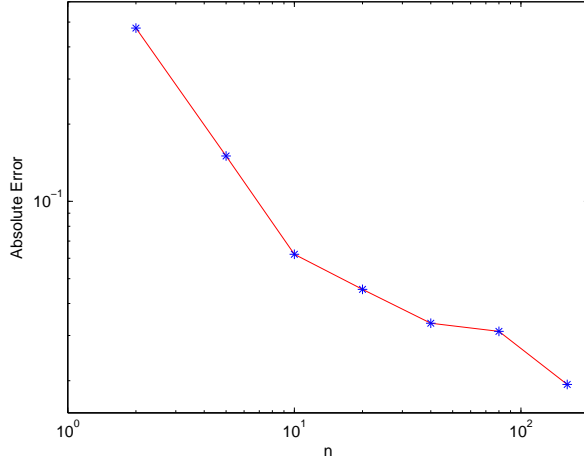


Figure 11: A 10-dimensional example with tridiagonal generator in Example 6.8.

In this case one may check straightforwardly that

$$[G_\gamma]_{ii} = 3 \quad \text{and} \quad [G_\gamma(t, x, y, z, \gamma)]_{ij} = -\frac{\gamma_{ij}}{(1 + \gamma_{ij}^2)^{\frac{3}{2}}}, \quad |i - j| = 1.$$

When $d = 10$, this example is out of the scope of our monotonicity Assumption 3.3. However, if we test it using $T = 0.2$, $x_0 = (1, 2, \dots, 10)$, the numerical results show that our scheme still converges to the true solution, $\sin(55) = -0.999755$, as presented in Figure 11.

We shall remark though that this example is computationally more expensive than Example 6.2 because here we need to approximate $3d - 2$ second derivatives. \square

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